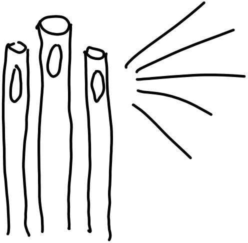
Compressed sensing and quantum state tomogrpahy

– recovering sparse objects from few measurements.



David Gross, Uni Hannover

QQQ Meeting
FU Berlin
October 2009

Outline

I will

- Motivate the problem,
- tell you our results,
- and then prove something else to you.

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The starting point

- Let ρ be an unknown density matrix on $\mathbb{C}^d$.
- Assume that rank $\rho = r$, with $r \ll d$.
- How many parameters are needed to specify ρ?
  - Well, roughly $r\,d$.
- How many coefficients to I need to know to recover ρ using full tomography?
  - Well, clearly $d^2 \gg rd$.
- What a waste.

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Central question

Can we obtain complete information about an unknown quantum state using substantially fewer than d^2 measurement settings, if that state is (essentially) low-rank?

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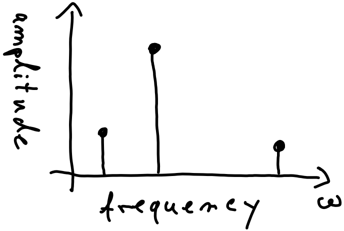
Yes, we can.

Classical analogue



- Spectrum of an organ: at any time few (r) out of many possible (d) frequencies sound.
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- Task: find that spectrum using few measurements.

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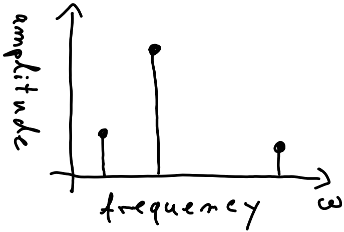

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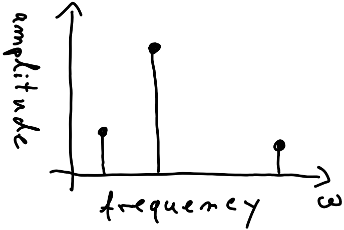
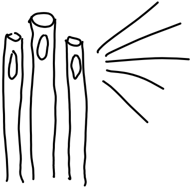
Classical analogue

- First idea: measure in frequency domain.
  - Fails. Need d sensors.
- Second idea: take few samples in time domain.
  - Nyquist would suggest that we need $f_{\text{max}}/2$ samples. Far too many.
  - But Nyquist holds for *general* signals — hope to optimize when support is low.





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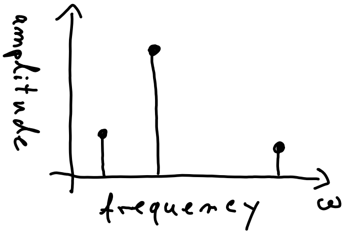

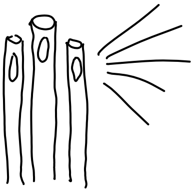

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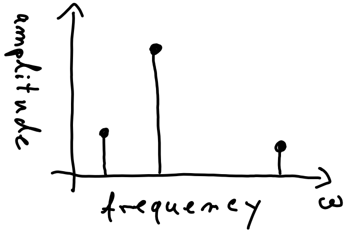

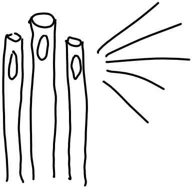

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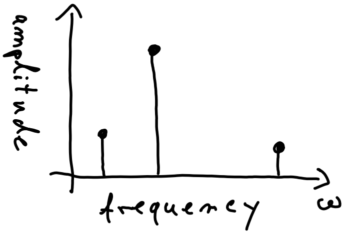

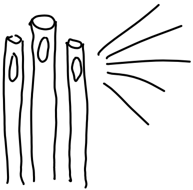

Classical compressed sensing

Theorem [Candès, Tao, *et.al.* (2004)]

- Knowing $r\,(\log d)$ randomly chosen time-domain amplitudes, one can recover any signal ψ composed from at most r frequencies.
- The scheme is probabilistic, and succeeds with high probability.
- The recovery is *exact*.
- It is achieved *computationally efficiently*. The signal uniquely solves the convex problem

$$\begin{aligned} \text{minimize}_{\psi'} \quad & \|\hat{\psi}'\|_{\ell_1} \\ \text{subject to} \quad & \psi'(t) = c_t \quad \forall\, c_t \in \Omega, \end{aligned}$$

with Ω the set of observed time-domain amplitudes

$$c_t = \psi(t).$$

Going quantum

Return to unknown rank-r density matrix ρ.

- Strong analogues to compressed sensing: want to learn *sparse object* without knowing the sparsity pattern.

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Indeed, previous results [Candès, Tao, *et.al.* (2008)] extend compressed sensing techniques to *matrix completion*.

- Task here: reconstruct low-rank matrix from few randomly chosen matrix elements.
- Not directly applicable to quantum case. More natural here: sample e.g. *Pauli expectation values*

$$\operatorname{tr} \rho \left(\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n} \right)$$

  for n-qubit system.
- Doesn't work for all low-rank matrices, but just for certain "incoherent ones".
- Proof *very* complicated (ca. 50 pages, written by Fields Medalist).

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Quantum compressed sensing

Theorem [Gross, Liu, Flammia, Becker, Eisert (2009)].

- Knowing $rd\,(\log d)$ randomly chosen Pauli expectation values, one can recover any rank-r density operator ρ.
- The scheme is probabilistic, and succeeds with high probability.
- The recovery is *exact*.
- It is achieved *computationally efficiently*. The density matrix ρ uniquely solves the convex problem

$$\begin{aligned} \text{minimize} \quad & \|\rho'\|_1 \\ \text{subject to} \quad & \operatorname{tr} \rho' \sigma_{\mathbf{i}} = c_{\mathbf{i}} \quad \forall\, \mathbf{i} \in \Omega, \end{aligned}$$

  with Ω the set of observed Pauli expectation values

$$c_{\mathbf{i}} = \operatorname{tr} \rho(\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}).$$

Details on blackboard

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